

Explicit factorization of Seiberg-Witten curves with matter from random matrix models.¹

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Abstract

Within the Dijkgraaf-Vafa correspondence, we study the complete factorization of the Seiberg-Witten curve for $U(N_c)$ gauge theory with $N_f < N_c$ massive flavors. We obtain explicit expressions, from random matrix theory, for the moduli, parametrizing the curve. These moduli characterize the submanifold of the Coulomb branch where all monopoles become massless. We find that the matrix model reveals some nontrivial structures of the gauge theory. In particular the moduli are additive with respect to adding extra matter and increasing the number of colors.

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1 Introduction

Very recently there appeared new powerful methods of extracting effective superpotentials for a wide class of $\mathcal{N} = 1$ gauge theories. The proposal by Dijkgraaf and Vafa [1, 2], building upon earlier string theoretical constructions [3, 4], links these superpotentials with quantities in random matrix models. The proposal have since been proven [5, 6]. The proposal has been extended to theories with fundamental matter [7] and there has been significant work on studying some features of the link with Seiberg-Witten curves [8, 9, 10, 11, 12, 13, 14].

These curves made their appearance in the ground-breaking work of Seiberg and Witten [15, 16] on the study of $\mathcal{N} = 2$ supersymmetric gauge theories. It turned out that one can describe the low energy dynamics of the gauge theory in terms of geometrical properties of the Seiberg-Witten curves. Subsequently it was realized [16, 19] that one could also study, within the same framework, deformations to $\mathcal{N} = 1$ theories by adding a tree level superpotential. In this context one is led to the points (submanifolds) in moduli space where monopoles become massless and condense. At these points the Seiberg-Witten curve factorizes – it has only two single zeroes (branch points) in the case of complete factorization, where all monopoles condense. Once the form of the Seiberg-Witten curve at the factorization point is known one can calculate effective potentials for the deformed theory [16, 19, 4].

The possibility of integrating in the glueball field S was realized in [9] and used to check the matrix model result following from the Dijkgraaf-Vafa proposal for deformed pure $\mathcal{N} = 2$ theories (without fundamental matter). There the explicit factorization of curves without matter of [19] was known. Factorization properties were also used in the context of $SO(N_c)$ theories [20] and multi-trace operators [21] to obtain the appropriate effective superpotentials $W_{eff}(S)$.

However once one adds fundamental matter to the theory, the factorization problem becomes exceedingly complicated (even in the case of $U(2)$ with 1 flavour) and no explicit solutions are known. The aim of this paper is to use the Dijkgraaf-Vafa proposal linking superpotentials to matrix models and derive, from the random matrix model solution, explicit factorization of Seiberg-Witten curves of $U(N_c)$ theory with $N_f < N_c$ fundamental matter fields.

If it were not for the Dijkgraaf-Vafa correspondence, we would not expect that any analytical solutions to this problem could exist. The fact that they can be obtained in this way shows that the random matrix model *with matter* is intimately linked to fine details of the geometry of the appropriate Seiberg-Witten curves.

In addition we find that the matrix theory variables capture a surprising robust structure of the factorized Seiberg-Witten curve. The ‘pure’ $\mathcal{N} = 2$ solutions and the new contributions of each flavor appear additively. All nonlinearity is concentrated in a single relation involving the scale of the gauge theory Λ and the matrix theory variables. Moreover we find that the ‘integrating-out’ equations of *pure* $\mathcal{N} = 2$ theories reappear, in terms of matrix variables, in the context of $\mathcal{N} = 2$ theories with fundamental flavors.

This arises naturally in the matrix model but is rather unexpected from a gauge theory perspective.

In section 7 we collect the final results of the paper.

The plan of this paper is as follows. In section 2 we describe the field theoretical ingredients in more detail. In section 3 we introduce the setup for the Dijkgraaf-Vafa proposal with matter fields. We then go to use the orthogonal polynomial method to rederive directly the solution of the factorization for pure $\mathcal{N} = 2$ theory, which is an ingredient of the expressions for theories with matter. In sections 5 and 6 we derive the linearity in couplings of the superpotential and the expressions for factorization. Section 7 contains the main result of the paper. In section 8 we give some specific examples and close the paper with a discussion. Three appendices contain some technical details.

2 Field theory considerations

It has been known for a long time that $\mathcal{N} = 2$ supersymmetric $U(N_c)$ gauge theories with $N_f < N_c$ massive flavors has a Coulomb branch that is not lifted by quantum corrections [15]. This quantum moduli space is N_c -dimensional, parametrized for example by $u_k = \left\langle \frac{1}{k} \text{tr } \Phi^k \right\rangle$ with $k \leq N_c$. At each point of the moduli space, the low energy theory is described by an $\mathcal{N} = 2$ effective abelian $U(1)^{N_c}$ gauge theory. All the relevant quantum corrections in the IR can be recast in terms of the period matrix of a particular meromorphic one-form of the auxilliary complex curve — the Seiberg-Witten curve [15, 15, 17, 18], or more precisely a family of genus $N_c - 1$ hyperelliptic Riemann surfaces:

$$y^2 = P_{N_c}(x, u_k)^2 - 4\Lambda^{2N_c - N_f} \prod_{i=1}^{N_f} (x + m_i), \quad (1)$$

with

$$P_{N_c}(x, u_k) = \langle \det(xI - \Phi) \rangle = \sum_{\alpha=0}^{N_c} s_{\alpha} x^{N_c - \alpha}. \quad (2)$$

The coefficients s_{α} are polynomials of the u_k 's parameterizing the Coulomb branch:

$$\alpha s_{\alpha} + \sum_{k=0}^{\alpha} k s_{\alpha-k} u_k = 0 \quad (3)$$

$$s_0 = 1, \quad u_0 = 0 \quad (4)$$

One can deform this $\mathcal{N} = 2$ theory to a $\mathcal{N} = 1$ gauge theory by adding a tree level superpotential:

$$W_{tree} = \sum_{p=1}^{N_c+1} g_p \cdot \frac{1}{p} \text{tr } \Phi^p. \quad (5)$$

The classical vacuum structure is given by all possible distributions of the N_c eigenvalues ϕ_k of Φ amongst the N_c critical points a_j of the potential. This corresponds, at the classical level, to a breaking of the $U(N_c)$ gauge symmetry to the gauge group $\prod_{i=1}^{N_c-n} U(N_i)$ with $\sum_{i=1}^{N_c-n} N_i = N_c$, where the N_i 's are the nonzero multiplicities of the eigenvalues. For the purposes of this paper we can safely set $g_{N_c+1} = 0$ as we will be considering the case with no breaking of gauge symmetry (complete factorization).

Turning to the quantum picture, the presence of this superpotential will lift the quantum moduli space, characteristic of the $\mathcal{N} = 2$ Coulomb phase, except for the codimension n submanifolds, where n mutually local magnetic monopoles become massless. These are the $\mathcal{N} = 1$ vacua solving the F-flatness and D-flatness conditions and are characterized by a monopole condensate of the massless monopoles. This is believed to produce, by the dual Meissner effect, the expected confinement of the electric $\mathcal{N} = 1$ theory. Hence, the final quantum theory is described at low energies by a $\mathcal{N} = 1$ $U(1)^{N_c-n}$ gauge theory. These $U(1)$'s can be thought of as the $U(1) \subset U(N_i)$ of the classical theory. The $\mathcal{N} = 1$ $SU(N_i)$ part of the theory confines, has a mass gap and is characterized by a gaugino condensate.

These $\mathcal{N} = 1$ vacua, being the codimensional n submanifolds of the $\mathcal{N} = 2$ Coulomb branch, where n mutually local monopoles become massless, are parameterized by the sets of moduli $\{u_k^{fact.}\}$ where the Seiberg-Witten curve factorizes, i.e. the r.h.s. of (1) has n double roots⁴ and $2(N_c - n)$ single roots:

$$P_{N_c}(x, u_k^{fact.})^2 - 4\Lambda^{2N_c-N_f} \prod_{i=1}^{N_f} (x + m_i) = F_{2(N_c-n)}(x) H_n^2(x). \quad (6)$$

Moreover it is shown in [8] that the reduced curve,

$$y^2 = F_{2(N_c-n)} \quad (7)$$

captures the full quantum dynamics of the $\mathcal{N} = 1$ $U(1)^{N_c-n}$ low energy theory.

⁴For simplicity we are assuming that higher order roots are not occurring.

The effective superpotential of the theory deformed by (5), where we set $g_{N_c+1} = 0$, is obtained by plugging the solutions $u_k^{fact.}$, parametrized by $N_c - n$ parameters into the tree level superpotential:

$$W_{eff} = \sum_{p=1}^{N_c} g_p u_p^{fact.}, \quad (8)$$

Then (8) should be minimized with respect to the $N_c - n$ parameters.

Complete factorization

In this paper we will be interested in the case where $N_c - 1$ mutually local monopoles condense. This corresponds to a complete factorization of the Seiberg-Witten curve: the vacua form a 1 dimensional submanifold such that the curve has only 2 single roots and $N_c - 1$ double roots:

$$P_{N_c}(x, u_k^{fact.})^2 - 4\Lambda^{2N_c - N_f} \prod_{i=1}^{N_f} (x + m_i) = (x - a)(x - b)H_{N_c-1}^2(x) \quad (9)$$

The main goal of this paper is to find explicit expressions for the moduli $u_k^{fact.}$ where complete factorization occurs. Let us briefly review the solution for the *pure* $\mathcal{N} = 2$ Yang-Mills case found by Douglas and Shenker [19] some time ago using Chebyshev polynomials.

Their solution factorizing the curve $P_{N_c}(x, u_k)^2 - 4\Lambda^{2N_c}$ is⁵

$$u_p^{fact.}(\Lambda, u_1) = \frac{N_c}{p} \sum_{q=0}^{[p/2]} \binom{p}{2q} \binom{2q}{q} \Lambda^{2q} \left(\frac{u_1}{N_c}\right)^{p-2q}. \quad (10)$$

Note that the one dimensional submanifold where the $U(N_c)$ curve completely factorized, is parametrized by $u_1 = \text{tr } \Phi$. These results can be easily restricted to the $SU(N_c)$ case, by putting explicitly $u_1 = 0$. All parameters are then uniquely fixed in terms of the only scale of the theory Λ .

Integrating in S

The quantum $\mathcal{N} = 1$ effective potential generated by the tree level potential is obtained by minimizing (8)

$$W_{eff}(\Lambda, u_1) = \sum_{p=1}^{N_c} g_p u_p^{fact.}(\Lambda, u_1) \quad (11)$$

⁵Adapted to the $U(N_c)$ gauge group [9].

with respect to u_1 .

Here we follow an alternative route taken in [9], and integrate in S by performing a Legendre transformation with respect to $\log \Lambda^{2N_c - N_f}$. The superpotential is then given by:

$$\begin{aligned} W_{eff}(S, u_1, \Omega, \Lambda) &= S \log \Lambda^{2N_c - N_f} + W_{eff}(S, u_1, \Omega) = \\ &= S \log \Lambda^{2N_c - N_f} - S \log \Omega^{2N_c - N_f} + \sum_{p=1}^{N_c} g_p u_p^{fact.}(\Omega, u_1) \end{aligned} \quad (12)$$

Note that the only Λ -dependence is in the linear $S \log \Lambda^{2N_c - N_f}$ term. Integrating out S forces $\Omega = \Lambda$ and brings us back to (11). In order to get the effective potential $W_{eff}(\Lambda, S)$ one has to integrate out both Ω and u_1 .

Our strategy for identifying the factorization parameters $u_p^{fact.}$ for the theory with matter is to rewrite the random matrix expression in the form given by the last two terms of (12) and to read off the appropriate gauge theoretic moduli $u_p^{fact.}$. The first term in (12) could also be absorbed into the matrix model expressions if appropriate rescalings of the integration measure of the matrix model was made. We will not do this here.

For later reference let us quote the equations of motion for the pure $\mathcal{N} = 2$ gauge theory:

$$\frac{\partial W_{eff}(S, u_1, \Omega)}{\partial \log \Omega^{2N_c}} = \sum_{p \geq 2} g_p \sum_{q=0}^{[p/2]} \frac{q}{p} \binom{p}{2q} \binom{2q}{q} \Omega^{2q} \left(\frac{u_1}{N_c} \right)^{p-2q} - S = 0 \quad (13)$$

$$\frac{\partial W_{eff}(S, u_1, \Omega)}{\partial u_1} = \sum_{p \geq 1} g_p \sum_{q=0}^{[p/2]} \frac{p-2q}{p} \binom{p}{2q} \binom{2q}{q} \Omega^{2q} \left(\frac{u_1}{N_c} \right)^{p-2q-1} = 0 \quad (14)$$

These follow directly from (12) and (10).

3 The Dijkgraaf-Vafa proposal with fundamental matter

According to the Dijkgraaf-Vafa prescription the the perturbative part of the superpotential can be expressed as [1]

$$W_{eff}(S) = N_c \frac{\partial \mathcal{F}_{\chi=2}(S)}{\partial S} + \mathcal{F}_{\chi=1}(S) \quad (15)$$

where the \mathcal{F}_χ are defined through the matrix integral [7, 1]

$$e^{-\frac{N^2}{S^2}\mathcal{F}_{\chi=2}(S)-\frac{N}{S}\mathcal{F}_{\chi=1}(S)+\dots} = \int D\Phi DQ_i D\tilde{Q}_i e^{-\frac{N}{S}(\text{tr } V(\Phi) - mQ_i\tilde{Q}_i - Q_i\Phi\tilde{Q}_i)} \quad (16)$$

where we included the coupling of fundamental matter to the adjoint field in accordance with $\mathcal{N} = 2$ supersymmetry.

The matter fields in the matrix model (16) appear only quadratically and hence may be integrated out giving

$$\int D\Phi \det(m + \Phi)^{-1} e^{-\frac{N}{S}\text{tr } V(\Phi)} = Z \cdot \langle \det(m + \Phi)^{-1} \rangle \quad (17)$$

where Z is the partition function of the matrix model without matter. For the complex matrix model it is well known that Z will contribute only to $\mathcal{F}_{\chi=2}$ and not to $\mathcal{F}_{\chi=1}$. Using large N factorization we also have

$$\langle \det(m + \Phi)^{-1} \rangle = \langle e^{-\log \det(m + \Phi)} \rangle = \langle e^{-\text{tr } \log(m + \Phi)} \rangle = e^{-\langle \text{tr } \log(m + \Phi) \rangle} \quad (18)$$

We see that the matter determinant appears at subleading order in N w.r.t the tree level potential. This has the important consequence that the saddle point solution (eigenvalue density $\rho(\lambda)$) and hence $\mathcal{F}_{\chi=2}$ will *not be influenced* by the presence of matter. The $\chi = 1$ contribution is then given by

$$\mathcal{F}_{\chi=1} = \sum_{i=1}^{N_f} \int d\lambda \rho(\lambda) \log(m_i + \lambda) \quad (19)$$

In the following section we will evaluate the $\mathcal{F}_{\chi=2}$ piece of the partition function Z appearing in (17) using the method of orthogonal polynomials. As a byproduct we will rederive the factorization expression for the pure gauge theory case. Then in section 5 we will start investigating the matter contribution (19).

4 Orthogonal polynomials and $\mathcal{F}_{\chi=2}$

In this section we will use the method of the orthogonal polynomials to study thoroughly the one cut solution. A very nice introduction to this powerful method can be found in [22]. We will show that all results obtained from the field theory analysis appear naturally in this setting from random matrix computations (compare [9]).

The orthogonal polynomials associated to the matrix model with potential $\frac{N}{S} \text{tr} V(\Phi)$ satisfy the recursion relation

$$sP_n(s) = P_{n+1}(s) + T_n P_n(s) + R_n P_{n-1}(s) \quad (20)$$

In the large N limit the recursion coefficients T_n, R_n can be taken to be continuous functions of $u = S \cdot (n/N) \equiv S \cdot x$. In addition, the equations of motion of the matrix model with general potential $W_{tree}(\Phi)$ become purely algebraic [23, 22]:

$$\int \frac{dz}{2\pi i} V' \left(z + \frac{R(Sx)}{z} + T(Sx) \right) = Sx \quad (21)$$

$$\int \frac{dz}{2\pi i} \frac{1}{z} V' \left(z + \frac{R(Sx)}{z} + T(Sx) \right) = 0 \quad (22)$$

When $x = 1$ we will denote $R(S)$ by R and $T(S)$ by T . These two variables are related to the endpoints a and b of the support of the matrix eigenvalue distribution through

$$T = \frac{a+b}{2} \quad (23)$$

$$R = \frac{(a-b)^2}{16}. \quad (24)$$

For a general matrix potential $V(\Phi) = \sum g_p \frac{1}{p} \Phi^p$, one obtains easily:

$$u \equiv Sx = \sum_{p \geq 2} g_p \sum_{q=0}^{[p/2]} \frac{q}{p} \binom{p}{2q} \binom{2q}{q} R(Sx)^q T(Sx)^{p-2q} \quad (25)$$

$$v \equiv \sum_{p \geq 1} g_p \sum_{q=0}^{[p/2]} \frac{p-2q}{p} \binom{p}{2q} \binom{2q}{q} R(Sx)^q T(Sx)^{p-2q-1} = 0, \quad (26)$$

defining u and v for later convenience⁶. In terms of these polynomials, the matrix model partition function takes a very elegant form:

$$Z = N! h_0^N e^{N^2 \int_0^1 dx (1-x) \ln R(Sx)}. \quad (27)$$

where h_0 is the integral

$$h_0 = \int_{-\infty}^{\infty} ds e^{-\frac{N}{S} V(s)} \quad (28)$$

⁶Apart from the present section u and v are always taken at $x = 1$.

From equation (27), one can easily extract the $\chi = 2$ contribution (we will comment on the h_0 piece below)

$$\mathcal{F}_{\chi=2} = -S^2 \int_0^1 dx (1-x) \ln R(Sx) \quad (29)$$

Performing an integration by parts leads to:

$$-S \int_0^S du \ln R(u) + \int_0^S du u \ln R(u) \quad (30)$$

Following the Dijkgraaf-Vafa prescription, the contribution to the gauge theory effective potential is proportional to:

$$\frac{\partial \mathcal{F}_{\chi=2}}{\partial S} = -S \ln R(S) + \int_{R(0)=0}^{R(S)} \frac{u(R, T)}{R} dR \quad (31)$$

The remaining integral is rather tricky. The integrand is a function of two variables R and T , tied together by the highly nonlinear constraint $v = 0$. This complicates the integral, and moreover spoils the manifest linearity in the couplings g_p , appearing in the matrix potential. It is convenient, however, to treat the one-dimensional integral as an integral of a 1-form over the path $\{v = 0\}$ in the two dimensional R - T plane. Then one can rewrite the integral as an integral over a *closed* 1-form ω , such that the original integral remains unchanged. One can then deform the contour $v = 0$ into a piece with $R = 0$, while integrating T from $T(0)$ to $T(S)$ and another piece, keeping T fixed at $T(S)$, and integrating R from 0 to $R(S)$ (see fig. 1):

$$\int_{R(0)=0}^{R(S)} \frac{u(R, T)}{R} dR \equiv \int_{v=0} \frac{u(R, T)}{R} dR = \int_{v=0} \omega = \int_{T=const} \omega + \int_{R=const} \omega \quad (32)$$

An extension of the integrand of (31) to a closed one-form can be found to be

$$\omega = \frac{u}{R} dR + v dT. \quad (33)$$

Note that the extra piece doesn't give a contribution to the original integral, which is taken over $v = 0$.

Performing the integral is now straightforward. The first piece, integrating over T and constraining R to 0 gives:

$$\int_{T(0)}^{T(S)} v dT = \sum_{p \geq 1} \frac{g_p}{p} (T^p(S) - T^p(0)). \quad (34)$$

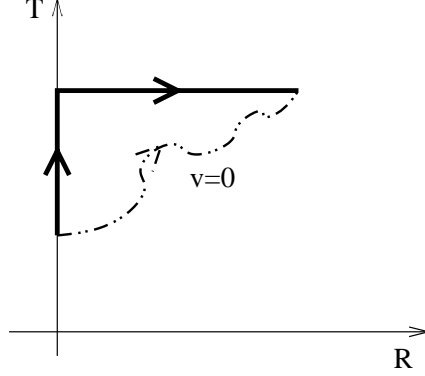


Figure 1: The dotted line represents the original contour of integration, while the solid lines show the deformed contour in the T - R plane.

The evaluation of the integral at $T(0)$, can be cancelled (at the saddle point) against the h_0^N , appearing in the partition function (27). For the second piece, one fixes T at $T(S)$, and integrates over R :

$$\int_0^{R(S)} \frac{u}{R} dR = \sum_{p \geq 2} \frac{g_p}{p} \sum_{q=1}^{[p/2]} \binom{p}{2q} \binom{2q}{q} R^q T^{p-2q} \quad (35)$$

From now on until the end of the paper we will denote by R and T the recursion coefficients defined at S . Bringing the two contributions together, leads to the following result:

$$W_{eff} = -N_c S \ln R + N_c \sum_{p \geq 1} \frac{g_p}{p} \sum_{q=0}^{[p/2]} \binom{p}{2q} \binom{2q}{q} R^q T^{p-2q} \quad (36)$$

This is exactly, term by term, the field theory result for *pure* $\mathcal{N} = 2$ $U(N_c)$ gauge theory ((12) and (10)), provided we identify field theory variables and matrix theory variables in the following way:

$$R = \Omega^2, \quad T = \frac{u_1}{N_c} \quad (37)$$

The first identification is extracted from the linear term in S , the second part follows from the term linear in g_1 . As it should, the equations of motion for R and T (25)-(26), are mapped, under this identification, to the equations obtained from integrating out Ω and u_1 (13)-(14).

From (36) we may read off the coefficients of g_p :

$$\mathcal{U}_p^{pure}(R, T) = \frac{1}{p} \sum_{q=0}^{[p/2]} \binom{p}{2q} \binom{2q}{q} R^q T^{p-2q} \quad (38)$$

Under the identification (37) and setting $\Omega = \Lambda$ these coincide exactly with the factorization solution (10) of Douglas and Shenker for pure $\mathcal{N} = 2$ gauge theory. For theories with matter, (38) will be just a part of the final result, and an identification between R , T and gauge theoretic quantities will have to be made only after the $\mathcal{F}_{\chi=1}$ contribution is evaluated in the following sections.

5 The $\mathcal{F}_{\chi=1}$ matter contribution

In the previous section we have rederived the result that the first piece in (15) can be recast in the form

$$N_c \frac{\partial \mathcal{F}_{\chi=2}}{\partial S} = N_c \left[-S \log R + \sum g_p \mathcal{U}_p^{pure}(R, T) \right] \quad (39)$$

and the equations of motion for R and T derived from (39) are exactly the random matrix saddle point equations

$$S = u \equiv \sum_{p \geq 2} g_p \sum_{q=0}^{[p/2]} \frac{q}{p} \binom{p}{2q} \binom{2q}{q} R^q T^{p-2q} \quad (40)$$

$$0 = v \equiv \sum_{p \geq 1} g_p \sum_{q=0}^{[p/2]} \frac{p-2q}{p} \binom{p}{2q} \binom{2q}{q} R^q T^{p-2q-1} \quad (41)$$

The first of these can be interpreted as the formula for integrating in S . The linearity of (39) in the random matrix couplings is essential for identifying the $\mathcal{U}_p^{pure}(R, T)$ with the point in $\mathcal{N} = 2$ moduli space where the Seiberg-Witten curve $y^2 = P_{N_c}(x, u_k)^2 - 4\Lambda^{2N_c}$ factorizes.

In the following section we will recast the random matrix expression (19) for $\mathcal{F}_{\chi=1}$ in the same way:

$$\mathcal{F}_{\chi=1} = \sum_{i=1}^{N_f} \left[S \log L(R, T, m_i) + \sum g_p \mathcal{U}_p^{matter}(R, T, m_i) \right] \quad (42)$$

Let us comment on an ambiguity, leading to a unique determination of a crucial term to obtain the correct final result. Once we start interpret this expression for $\mathcal{F}_{\chi=1}$, as a part of the effective gauge theory potential, we should be able to integrate out R and T . The equations of motion thus obtained, should be consistent with the random matrix saddle point equations. But, as explained in section 3, adding fundamental matter in the gauge theory does not modify the saddle point equations of the matrix model. So it is necessary to impose that the equations of motion for R and T , derived from the effective potential, with the $\mathcal{F}_{\chi=1}$ contribution are consistent with (40) and (41). This fact is highly non-trivial and reveals an unexpected relation between the factorization of Seiberg-Witten curves with and without matter.

We will show that the addition of $\mathcal{F}_{\chi=1}$ to the superpotential does not change the equations of motion for R and T provided we add an extra term proportional to v . From the matrix model perspective nothing changes, while the extra term turns out to be crucial to obtain the correct gauge theoretical interpretation of our results. We stress that this does not alter the Dijkgraaf-Vafa prescription. It merely solves an ambiguity that arises when interpreting the matrix model variables directly in terms of gauge theoretical quantities.

Let us now focus on a single summand of (19). In this case it seems to be difficult to use the orthogonal polynomial method, which was the most straightforward way of deriving the $\mathcal{F}_{\chi=2}$ contribution. Here it is more convenient to use the saddle point expression for the eigenvalue density of the single-cut solution:

$$\rho(x) = \frac{1}{2\pi} M(x) \cdot \sqrt{(b-x)(x-a)} \quad (43)$$

where $M(x)$ is a polynomial which is expressible in terms of the random matrix potential through

$$M(x) = \int_{C_\infty} \frac{dw}{2\pi i} \frac{V'(w)}{(w-x)\sqrt{(w-a)(w-b)}} \quad (44)$$

The features of interest of the above expression are (i) it is *linear* in the explicit dependence on the couplings g_p , (ii) the coefficients of g_p are universal functions of the endpoints a , b and hence of the variables R and T , (iii) for given g_p , its coefficient is a polynomial in x of order $p-2$.

In order to obtain an explicit dependence on R and T let us perform the

change of variables

$$x = \frac{1}{2}(a + b) + \frac{b - a}{2}\psi \equiv T + 2\sqrt{R}\psi \quad (45)$$

Then the contribution to $\mathcal{F}_{\chi=1}$ of a single flavour is given by

$$R \cdot \frac{2}{\pi} \int_{-1}^1 d\psi \sqrt{1 - \psi^2} \cdot M(\psi) \cdot \log(m + T + 2\sqrt{R}\psi) + v f(R, T) \quad (46)$$

As noted before, one should take into account a possible addition of v multiplied by any function $f(R, T)$ (since $v = 0$ is an equation of motion of the random matrix model).

6 Factorization formulas

We will first derive the formulas involving g_1 and g_2 , in particular this will allow us to fix uniquely the function $f(R, T)$. Also this will make the general structure more transparent. Then we will derive the results for arbitrary g_p .

Contribution of g_1, g_2

To this order $M(\psi) = g_2$ and the formula (46) gives

$$g_2 R \left[\log(m + T) + \frac{2}{\pi} \int_{-1}^1 d\psi \sqrt{1 - \psi^2} \log \left(1 + 2 \frac{\sqrt{R}}{m + T} \psi \right) \right] \quad (47)$$

This can be evaluated to give

$$g_2 R \left[\log \left(\frac{m + T + \sqrt{(m + T)^2 - 4R}}{2} \right) + (m + T) \frac{m + T - \sqrt{(m + T)^2 - 4R}}{4R} - \frac{1}{2} \right] \quad (48)$$

At this stage we should identify the coefficient of the logarithm with S (here this is trivial since to this order the equation of motion for R is just $S = g_2 R$, but later we will see that this property will hold in general). Thus we are left with

$$S \log \left(\frac{m + T + \sqrt{(m + T)^2 - 4R}}{2} \right) + g_2 \left[\frac{m + T}{4} (m + T - \sqrt{(m + T)^2 - 4R}) - \frac{R}{2} \right] \quad (49)$$

which indeed has the form of (42). Now we require that the saddle point equations remain consistent with the $\mathcal{F}_{\chi=1}$ equations of motion. This determines uniquely the term

$$v(g_1, g_2, R, T) \cdot f(R, T) = (g_1 + g_2 T) \cdot f(R, T) \quad (50)$$

Indeed the requirement that integrating out T from the sum of (49) and (50) gives $v = g_1 + g_2 T = 0$ fixes $f(R, T)$ uniquely to be

$$f(R, T) = -\frac{1}{2} \left(m + T - \sqrt{(m + T)^2 - 4R} \right) \quad (51)$$

This extra term does not change the equation for R , which is consistent with the saddle point equations. This function will stay unchanged in the general case. We may now read off the final expressions for the mass dependent contributions to u_1 and u_2 :

$$\mathcal{U}_1^{matter}(R, T, m) = -\frac{1}{2} \left(m + T - \sqrt{(m + T)^2 - 4R} \right) \quad (52)$$

$$\begin{aligned} \mathcal{U}_2^{matter}(R, T, m) = & \frac{m + T}{4} (m + T - \sqrt{(m + T)^2 - 4R}) - \frac{R}{2} + \\ & -\frac{T}{2} \left(m + T - \sqrt{(m + T)^2 - 4R} \right) \end{aligned} \quad (53)$$

Arbitrary g_p

We will now extend the previous considerations to the calculation of arbitrary \mathcal{U}_p^{matter} . The general structure will remain unchanged. The function $f(R, T)$ multiplying v will remain unmodified (as it should). Also the coefficient of the logarithm will turn out to be exactly S .

In appendix A we derive the following expression for the polynomial $M(\psi)$:

$$M(\psi) = \sum_{p \geq 2} g_p \sum_{n=0}^{p-2} c_{p,n} \psi^n \quad (54)$$

where

$$c_{p,n} = 2^n R^{\frac{n}{2}} \sum_{k=0}^{\left\lfloor \frac{p-n-2}{2} \right\rfloor} \binom{2k}{k} \binom{p-1}{2k+n+1} R^k T^{p-n-2-2k} \quad (55)$$

So the integral (46) can be rewritten as

$$\begin{aligned} \sum_{p \geq 2} g_p \sum_{n=0}^{p-2} c_{p,n} R \left[\log(m+T) \cdot \frac{2}{\pi} \int_{-1}^1 d\psi \sqrt{1-\psi^2} \cdot \psi^n + \right. \\ \left. + \frac{2}{\pi} \int_{-1}^1 d\psi \sqrt{1-\psi^2} \cdot \psi^n \cdot \log \left(1 + 2 \frac{\sqrt{R}}{m+T} \psi \right) \right] \end{aligned} \quad (56)$$

We now have to distinguish two cases.

n odd: Then the first integral vanishes and we will denote the second integral by $f_n(z)$. As discussed in appendix B, $f_n(z)$ is essentially a polynomial in z of order $(n+1)/2$ (divided by $(z-1)^{n/2+1}$) when expressed in terms of the variable

$$z = \frac{m+T}{2R} \left(m+T + \sqrt{(m+T)^2 - 4R} \right) \quad (57)$$

Appendix B contains a general formula for $f_n(z)$. Explicit expressions for some specific cases are shown in table 1 in section 7.

n even: In this case the first integral is nonvanishing. Moreover the second integral involves a logarithm with the same coefficient as $\log(m+T)$ in the first integral. Together they combine to give

$$\left\{ \sum_{p \geq 2} g_p \sum_{l=0}^{\lfloor \frac{p-2}{2} \rfloor} c_{p,2l} R \cdot \frac{2^{-2l}}{l+1} \binom{2l}{l} \right\} \cdot \log \left(\frac{m+T + \sqrt{(m+T)^2 - 4R}}{2} \right) \quad (58)$$

It is shown in appendix C that the coefficient in curly braces is exactly equal to S . The second integral with the logarithmic part subtracted out has again a simple polynomial structure (see appendix B for details).

At this stage we arrive to the analogue of (49):

$$S \log \left(\frac{m+T + \sqrt{(m+T)^2 - 4R}}{2} \right) + \sum_{p \geq 2} g_p \left[\sum_{n=0}^{p-2} c_{p,n} R f_n(z) \right] \quad (59)$$

Again we have to add to this the correction term

$$v \cdot f(R, T) \equiv - \sum_{p \geq 1} g_p v_p \frac{1}{2} \left(m+T - \sqrt{(m+T)^2 - 4R} \right) \quad (60)$$

We checked explicitly for some cases that this term together with (59) gives equations of motion consistent with the random matrix constraints.

The sum of (59) and (60) is now of the expected form (42), thus defining $\mathcal{U}_p^{\text{matter}}(R, T, m_i)$.

7 Final results

Putting all results together ((36), (59), (60)), and putting in the term $S \log \Lambda^{2N_c - N_f}$ as required from the Dijkgraaf-Vafa prescription gives a prediction from the matrix model side for the quantum effective gauge potential:

$$W_{eff}(S, T, R, \Lambda) = S \log \frac{\Lambda^{2N_c - N_f} \prod_{i=1}^{N_f} \frac{1}{2} \left(m_i + T + \sqrt{(m_i + T)^2 - 4R} \right)}{R^{N_c}} \\ + \sum_{p \geq 1} g_p \left[N_c \mathcal{U}_p^{pure}(R, T) + \sum_{i=1}^{N_f} \mathcal{U}_p^{matter}(R, T, m_i) \right]. \quad (61)$$

This expression should be compared with the potential $W_{eff}(S, u_1, \Omega, \Lambda)$ (see (12)), obtained from the field theory analysis. The relation between the parameters of the matrix model and the field theory is highly non-linear:

$$u_1 = N_c T - \frac{1}{2} \sum_{i=1}^{N_f} (m_i + T - \sqrt{(m_i + T)^2 - 4R}) \quad (62)$$

$$\Omega^{2N_c - N_f} = \frac{R^{N_c}}{\prod_{i=1}^{N_f} \frac{1}{2} \left(m_i + T + \sqrt{(m_i + T)^2 - 4R} \right)} \quad (63)$$

In order to obtain the final factorization formulae we integrate out S from $W_{eff}(S, T, R, \Lambda)$:

$$\Omega^{2N_c - N_f} = \Lambda^{2N_c - N_f}, \quad (64)$$

Combined with (63) this gives an expression for Λ in terms of R and T . The remaining part of (61) should then be compared with field theory result:

$$W_{eff}(\Lambda, u_1) = \sum_{p \geq 1} g_p u_p^{fact.}(\Lambda, u_1), \quad (65)$$

where the $u_p^{fact.}$, is the one parameter solution, factorizing completely the Seiberg-Witten curve for $U(N_c)$ SYM with N_f flavours ($N_f < N_c$):

$$y^2 = P_{N_c}(x, u_k)^2 - 4\Lambda^{2N_c - N_f} \prod_{i=1}^{N_f} (x + m_i) \quad (66)$$

Comparing with the results from matrix models gives an expression for the $u_p^{fact.}$'s,

$$u_p^{fact.} = N_c \mathcal{U}_p^{pure}(R, T) + \sum_{i=1}^{N_f} \mathcal{U}_p^{matter}(R, T, m_i) \quad (67)$$

$$\begin{aligned}
f_0(z) &= \frac{1}{2(z-1)} & f_1(z) &= \frac{3z-4}{6(z-1)^{3/2}} \\
f_2(z) &= \frac{1}{16(z-1)^2} & f_3(z) &= \frac{30z^2-65z+32}{120(z-1)^{5/2}} \\
f_4(z) &= \frac{-3z^2+9z-5}{96(z-1)^3} & f_5(z) &= \frac{525z^3-1610z^2+1582z-512}{3360(z-1)^{7/2}} \\
f_6(z) &= \frac{-48z^3+168z^2-176z+59}{1536(z-1)^4} & f_7(z) &= \frac{4410z^4-17640z^3+25956z^2-16857z+4096}{40320(z-1)^{9/2}}
\end{aligned}$$

Table 1: Examples of the functions $f_n(z)$ for small n .

in terms of two parameters R and T , tied together with the constraint:

$$\Lambda^{2N_c-N_f} = \frac{R^{N_c}}{\prod_{i=1}^{N_f} \frac{1}{2} (m_i + T + \sqrt{(m_i + T)^2 - 4R})} \quad (68)$$

For completeness, we recall the formulas

$$\mathcal{U}_p^{pure}(R, T) = \frac{1}{p} \sum_{q=0}^{[p/2]} \binom{p}{2q} \binom{2q}{q} R^q T^{p-2q} \quad (69)$$

$$\mathcal{U}_1^{matter}(R, T, m) = -\frac{1}{2} \left(m + T - \sqrt{(m + T)^2 - 4R} \right) \quad (70)$$

$$\mathcal{U}_{p \geq 2}^{matter}(R, T, m) = \sum_{n=0}^{p-2} c_{p,n} R f_n(z) - v_p \frac{1}{2} \left(m + T - \sqrt{(m + T)^2 - 4R} \right) \quad (71)$$

In the above formula the coefficients $c_{p,n}$ are defined in (55), while v_p is just the coefficient of g_p in the constraint $v = 0$ (see eq. (41)). Finally the functions $f_n(z)$ are computed in appendix B and depend on the variable $z(R, T)$, given by (57). In table 1 we present the explicit forms of the functions $f_n(z)$ for $n \leq 7$.

In the final result (67) we see that both the N_c and N_f dependence is very simple. Moreover the contribution of each extra flavor enters additively the expression for the moduli. Another curious feature is the appearance of the original factorization solutions for the *pure* $\mathcal{N} = 2$ $U(N_c)$ gauge theory. Increasing the number of colors does not change the expressions for \mathcal{U}_p^{matter} in terms of R and T . However the only nontrivial change is encoded in the expression for Λ in terms of R and T .

If we wanted instead to obtain the effective potential $W(S, \Lambda)$, we would have to integrate out R and T from (61). On the field theory side it is very cumbersome how the structure of the Seiberg-Witten curve appears in the equations of motion for u_1 and Ω . On the matrix model, on the other

hand, the equations of motion for R and T appear naturally to be the same as ones obtained in the case without flavours. It seems that the particular combinations of u_1 and Ω , embodied in R and T , captures some nontrivial structure of the Seiberg-Witten curves.

8 Some examples

In this section we will study some examples to verify that the $u_p^{fact.}$ we obtained from random matrix models, do factorize the appropriate Seiberg-Witten curves with fundamental matter.

$U(2)$ with 1 flavour

In this case we have

$$u_1^{fact.} = 2T - \frac{1}{2} \left(m + T - \sqrt{(m+T)^2 - 4R} \right) \quad (72)$$

$$u_2^{fact.} = 2 \left(R + \frac{T^2}{2} \right) + \frac{1}{4} \left(m^2 - 2R - T^2 + (T-m)\sqrt{(m+T)^2 - 4R} \right) \quad (73)$$

$$\Lambda^3 = \frac{2R^2}{m + T + \sqrt{(m+T)^2 - 4R}} \quad (74)$$

We have verified that with the above choices, the discriminant of the Seiberg-Witten curve

$$y^2 = \left(x^2 - u_1 x - u_2 + \frac{1}{2} u_1^2 \right)^2 - 4\Lambda^3(x+m) \quad (75)$$

vanishes identically, which in the special case of 2 colours proves factorization. Note that for general N_c , complete factorization is a much stronger condition than the vanishing of the discriminant.

$SU(N_c)$ with 1 flavour

In order to consider $SU(N_c)$ theory we have to impose the constraint that $u_1 = 0$. Then the parameter T can be expressed in terms of R and the mass m of the additional flavour. Namely we have

$$u_1 \equiv N_c T - \frac{1}{2} \left(m + T - \sqrt{(m+T)^2 - 4R} \right) = 0 \quad (76)$$

which gives

$$T = \frac{m - \sqrt{m^2 - 4R + 4\frac{R}{N_c}}}{2(N_c - 1)} \quad (77)$$

Now R is linked directly to the scale of the gauge theory through

$$\Lambda^{2N_c-1} = \frac{R^{N_c}}{(m + T + \sqrt{(m + T)^2 - 4R})/2} \quad (78)$$

where the expression (77) should be used. The remaining formulas remain however quite complicated functions of R , m and N_c . This is in marked contrast to the case of pure gauge theory without fundamental matter where the passage from $U(N_c)$ to $SU(N_c)$ is very simple, and N_c enters linearly.

It is interesting to look at the $m \rightarrow \infty$ limit. Then $T \rightarrow 0$ as expected for a pure $\mathcal{N} = 2$ $SU(N_c)$ theory, while (78) becomes $\Lambda^{2N_c-1} = R^{N_c}/m$. Recall that in pure $SU(N_c)$ theory R had the interpretation of Λ_{pure}^2 . Hence we obtained the correct field theoretic matching of scales

$$\Lambda^{2N_c-1} m = \Lambda_{pure}^{2N_c} \quad (79)$$

Moreover it is easy to check that then the $\mathcal{U}_p^{matter}(R, T, m)$ give a vanishing contribution as the functions $f_n(z) \rightarrow 0$ when $z \rightarrow \infty$ (see appendix B). The above behaviour is quite clear from the Seiberg-Witten curve perspective. It is reassuring that it could be also obtained in a simple way from the random matrix formulas.

$U(7)$ with 3 flavours

As a final check of the formulas we considered $U(7)$ theory with 3 flavours with masses $m_1 = 1$, $m_2 = 2$, $m_3 = 3$. For the random choice of parameters $T = 0$ and $R = 0.2$ we find $\Lambda = 0.31643$, while the polynomial $P_7(x) = x^7 + 0.45018x^6 - 1.3447x^5 - 0.5382x^4 + 0.5048x^3 + 0.16048x^2 - 0.045x - 0.00704$. For the curve

$$y^2 = P_7(x)^2 - 4\Lambda^{11}(x+1)(x+2)(x+3) \quad (80)$$

we find single zeroes at $x = \pm 0.8944$ and a series of 6 double zeroes in between.

9 Discussion

In this paper we used the Dijkgraaf-Vafa proposal linking random matrix models with fundamental matter and superpotentials for obtaining explicit formulas for the complete factorization of Seiberg-Witten curves for $U(N_c)$ theories with $N_f < N_c$ flavours. These points in the moduli space, forming effectively a 1-parameter manifold, correspond to condensation of all species of monopoles. As a byproduct we obtained formulas for the solution of the random matrix model with matter with an arbitrary polynomial superpotential.

In order to identify the points in moduli space where the Seiberg-Witten curve factorizes we recast the random matrix solution in a way that exhibits (i) linearity in the couplings g_p of the deforming tree level superpotential, (ii) the whole dependence on the glueball superfield S could be written as a linear coupling of S to a logarithmic expression. The first property allowed us to identify the moduli space parameters of the factorized curve u_p as the coefficients of g_p , while the second property is exactly the one found in ‘integrating-in’ S and thus gave the expression for the gauge theoretic scale Λ in terms of random matrix model quantities.

The fact that the above procedure works is yet another argument for the Intriligator-Leigh-Seiberg linearity principle [24] and the validity of ‘integrating-in’. In addition it shows that the matrix model of the Dijkgraaf-Vafa proposal captures quite detailed properties of the field theoretical Seiberg-Witten curve. In fact we found it surprising that any analytical description of the very nonlinear complete factorization property could be found for the case with matter.

A curious feature of the random matrix formulas is that the solution for the u_p for the pure $\mathcal{N} = 2$ theory appears *linearly* in the complete expression for the theory with fundamental matter fields. The full ‘nonlinearity’ is encoded in the formula for Λ in terms of random matrix parameters. It would be interesting to understand this structure from the field-theoretical point of view. In addition the random matrix constraints expressed in terms of R and T don’t change when adding fundamental matter. They have precisely the form of equations of motion for the pure $\mathcal{N} = 2$ theory. But now the mapping between R and T and field theoretical Λ and u_1 becomes complicated. Nevertheless, the question why the pure $\mathcal{N} = 2$ equations still arise in a disguised form for theories with fundamental matter poses an interesting question from the gauge theory perspective.

There are numerous issues that one could investigate further. The factorization properties of the pure $\mathcal{N} = 2$ curve are linked with the concept of master field. This has been investigated in the context of associated random matrix theory in [25]. It would be interesting to study the factorization formulas obtained in this paper from a similar point of view, albeit it will surely be much more involved.

Another interesting question would be to explore the mathematical structure linking the random matrix model with matter with factorization properties of the associated curves. In this paper we relied heavily on recasting the random matrix expressions guided by *field theoretic* ingredients such as the ILS principle and integrating-in, for which there is no real direct proof. It would be very interesting to uncover the *mathematical* interrelation between such seemingly unconnected topics as the factorization of SW curves and random matrix models.

Finally we hope that the above results could be used for a more detailed investigation of the physics of $U(N_c)$ theories with $N_f < N_c$ flavours along the lines of [19, 26].

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A Formula for $M(\psi)$

Since we use explicitly the form of the eigenvalue density in the variable ψ , let us perform the changes of variables $x = T + 2\sqrt{R}\psi$, $w = T + 2\sqrt{R}\phi$ in the definition (44) of $M(x)$:

$$M(\psi) = \frac{1}{2\sqrt{R}} \text{Res}_{\phi=\infty} \frac{1}{\phi - \psi} \frac{V'(T + 2\sqrt{R}\phi)}{\sqrt{\phi^2 - 1}} \quad (81)$$

Using the power series expansion of the square root

$$\frac{1}{\sqrt{1 - \frac{1}{\phi^2}}} = \sum_{k=0}^{\infty} a_k \frac{1}{\phi^{2k}} \equiv \sum_{k=0}^{\infty} 2^{-2k} \binom{2k}{k} \frac{1}{\phi^{2k}} \quad (82)$$

it is straightforward to obtain the Laurent expansion of the function in (81):

$$\sum_{p \geq 2} g_p \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \psi^n \frac{1}{\phi^{2k+n+2}} a_k \sum_{l=0}^{p-1} \binom{p-1}{l} (2\sqrt{R})^{l-1} T^{p-1-l} \phi^l \quad (83)$$

From this expression we may isolate the coefficient of $1/\phi$ giving the result quoted in the text:

$$M(\psi) = \sum_{p \geq 2} g_p \sum_{n=0}^{p-2} c_{p,n} \psi^n \quad (84)$$

$$c_{p,n} = 2^n R^{\frac{n}{2}} \sum_{k=0}^{\lfloor \frac{p-n-2}{2} \rfloor} \binom{2k}{k} \binom{p-1}{2k+n+1} R^k T^{p-n-2-2k} \quad (85)$$

B Logarithmic integrals $f_n(z)$

Here we will derive the explicit form of the functions $f_n(z)$ related to the logarithmic integrals

$$I_n = \frac{2}{\pi} \int_{-1}^1 d\psi \sqrt{1-\psi^2} \cdot \psi^n \cdot \log(1+2x\psi) \quad (86)$$

where $x = \sqrt{R}/(m+T)$. In fact the results simplify significantly if one reexpresses everything in terms of the variable z

$$z = \frac{m+T}{2R} \left(m+T + \sqrt{(m+T)^2 - 4R} \right) \quad (87)$$

(x is expressed in terms of z as $\sqrt{z-1}/z$). We have to distinguish two cases:

n odd

The integral (86) can be performed using a series expansion of the logarithm, integrating it term by term using

$$\frac{2}{\pi} \int_{-1}^1 d\psi \sqrt{1-\psi^2} \cdot \psi^{2n} = \frac{2^{-2n+1}}{(2n+2)} \binom{2n}{n} \quad (88)$$

and resumming. The result is

$$f_n^{odd}(z) = \frac{2\sqrt{z-1}\Gamma\left(1+\frac{n}{2}\right)}{\sqrt{\pi}z\Gamma\left(\frac{5+n}{2}\right)} {}_3F_2\left(\frac{1}{2}, 1, 1+\frac{n}{2}; \frac{3}{2}, \frac{5+n}{2} \middle| \frac{4(z-1)}{z^2}\right) \quad (89)$$

For odd n , $f_n(z)$ is expressed through elementary functions (see examples in section 7). It has the form of a polynomial in z of order $(n+1)/2$ divided by $(z-1)^{\frac{n}{2}+1}$.

n even

The integral (86) can be again obtained using a resummation procedure. The result is

$$I_n^{even} = -\frac{2(z-1)\Gamma\left(\frac{3+n}{2}\right)}{\sqrt{\pi}z^2\Gamma\left(3+\frac{n}{2}\right)} {}_3F_2\left(1, 1, \frac{3+n}{2}; 2, 3+\frac{n}{2} \middle| \frac{4(z-1)}{z^2}\right) \quad (90)$$

In this case the integral (86) involves a logarithm, which together with the $\log(m+T)$ forms the logarithmic function multiplying S in (59). Thus to define the function $f_n(z)$ for even n we have to subtract from I_n^{even} this logarithm:

$$f_n^{even}(z) \equiv I_n^{even} - \frac{2^{-n+1}}{n+2} \binom{n}{\frac{n}{2}} \cdot \log\left(\frac{z-1}{z}\right) \quad (91)$$

Its general form turns out to be a polynomial of order $n/2$ divided by $(z-1)^{\frac{n}{2}+1}$. In table 1 in section 7, for completeness we present the explicit forms of the functions $f_n(z)$ for $n \leq 7$.

C Coefficient of the logarithmic term in (58)

In this appendix, we identify the coefficient of the logarithmic piece of the $\chi = 1$ contribution with S , as expected from field theory. From (46) one can easily read off the coefficient of $\log(m+T)$:

$$4R \int_{-1}^1 \frac{d\psi}{2\pi} \sqrt{1-\psi^2} M(\psi) \quad (92)$$

Using the elementary integral (88) one obtains the expression:

$$4R \sum_{p \geq 2} g_p \sum_{2l=0}^{p-2} \binom{2l}{l} c_{p,2l} \frac{2^{-2l-2}}{l+1} \quad (93)$$

Inserting the explicit expression for the coefficient $c_{p,2l}$, leads to:

$$\sum_{p \geq 2} g_p \sum_{l=0}^{\lfloor \frac{p-2}{2} \rfloor} \sum_{k=0}^{\lfloor \frac{p-2l-2}{2} \rfloor} \binom{2k}{k} \binom{2l}{l} \binom{p-1}{2l+2k+1} \frac{1}{l+1} R^{l+k+1} T^{p-2k-2l-2}. \quad (94)$$

Changing to a new summation variable $m = k + l + 1$ gives the expression:

$$\sum_{p \geq 2} g_p \sum_{m=1}^{\lfloor \frac{p}{2} \rfloor} \sum_{l=0}^{m-1} \frac{2m}{p(l+1)} \binom{p}{2m} \binom{2l}{l} \binom{2m-2l-2}{m-l-1} R^m T^{p-2m}. \quad (95)$$

This is exactly the expression for S , provided that:

$$2 \sum_{l=0}^{m-1} \frac{1}{l+1} \binom{2l}{l} \binom{2m-2l-2}{m-l-1} = \binom{2m}{m} \quad (96)$$

which can be verified.

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